Improved low-order model for shear flow driven by Rayleigh-Bénard convection

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(Received 31 May 1994)

An analysis of the low-order model for two-dimensional fluid flow with shear proposed by Drake *et al.* [Phys. Fluids B **4**, 488 (1992)] is undertaken. Their two-term model for the shear is an extension of the model put forth by Howard and Krisnamurti [J. Fluid Mech. **170**, 385 (1986)], and is shown to be an improved model in the sense that it respects certain conditions for vorticity conservation arising directly from the Boussinesq equations. In so doing, it provides a more realistic model of the physics involved. An important consequence of the improved model is the appearance of cutoff values for the shear instability that are dependent upon the aspect ratio of the interacting Rayleigh-Taylor cell. Numerical results are presented as confirmation of this prediction.

PACS number(s): 03.40.Gc, 47.20.Ky, 47.27.Pa

I. INTRODUCTION

The works of Howard and Krishnamurti have served as the foundation for the study of two-dimensional shear flow. In particular, the truncated Fourier model and its analysis presented in their paper "Large-scale flow in turbulent convection: a mathematical model" [1] is often implemented as the basis for low-order studies of the Boussinesq equations. Their model for the stream function and temperature perturbation begins with the three Fourier components that give rise to the Lorenz equations. These support a Rayleigh-Taylor (R-T) instability with single vertical (z) and horizontal (x) modes in a horizontally infinite channel of incompressible fluid with fixed boundaries top and bottom over which a constant temperature difference is maintained (Fig. 1). Based on their observation of a large scale horizontal flow witnessed during experimental work [2], Howard and Krishnamurti added a $\sin(z)$ term to the model of the stream function plus a complementary term to the stream function and temperature perturbation.

The authors placed a clear caveat on their work regarding the lack of higher-order R-T modes and the resulting disconnect between the model and real-world observations. However, a recent study of driven vortices [3,4] indicates that the Howard-Krishnamurti (H-K) model may be inappropriate for the study of even single mode phenomena due to the absence of a higher-order *shear* mode under certain nondegenerate parameter conditions, viz., for elongated (low aspect ratio) R-T modes. We will use the horizontal wave number for the lowest-order R-T mode being modeled, α, as the measure of aspect ratio $(2L_z/L_x)$.

The H-K model has served as a touchstone for recent work in two-dimensional magnetoconvection in the Boussinesq approximation. Building on the hydrodynamic simulations of Ginet and Sudan [5], Lantz [6] has performed extensive numerical studies for a conducting fluid in a horizontal magnetic field. Using an extension of the H-K equations which included magnetic fields, Lantz found his results predicting sheared convection compared favorably to the low-order model in the case where the Prandtl number ≈ 2 and $\alpha = 2$. Hughes and Proctor [7] have proposed a simplification to the sixth order H-K model justified in the limit of small Prandtl number and large α. Their resulting third order system has been utilized by Rucklidge and Mathews [8] and Mathews *et al.* [9] to map the bifurcations occurring in Boussinesq magnetoconvection in a vertical magnetic field.

The anomalous behavior of shear flow for low aspect ratio R-T modes was observed when Drake *et al.* [3] attempted to link certain tokamak plasma edge effects to the generation of shear flow. In that paper the additional shear term $\sin(3z)$ was introduced, justified by considering the requirement for vorticity conservation in the inviscid Boussinesq equations. Finn *et al.* [4] elaborated on and extended that work, showing that the shear in-

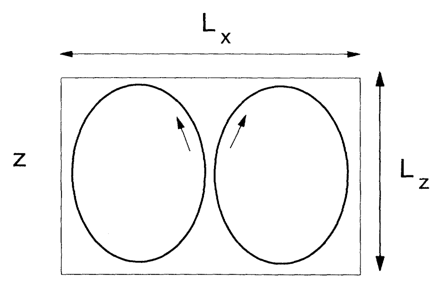

FIG. 1. Representation of a two-dimensional Rayleigh-Taylor convective cell, constrained by impenetrable walls in the vertical (z) direction between which a constant temperature difference is maintained and periodic boundary conditions in the horizontal (x) direction.

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1063-651X/95/51(1)/325(7)/$06.00

stability was either inviscid or viscous depending on the aspect ratio of the R-T cell.

In this paper, we will motivate the requirement for an additional shear mode term by deriving a statement of the conservation of vorticity directly from the vorticity equation in the Boussinesq approximation with no assumptions on viscosity. We will then demonstrate that the H-K model as originally proposed fails to meet this conservation condition for an inviscid fluid and for a viscous fluid in steady state. Next, the $\sin(3z)$ term will be introduced into the H-K model and we will shadow the original analysis of Howard and Krishnamurti up through the bifurcation leading to stable shear flow. The modified H-K system leads to the appearance of cutoff values for the shear mode which predict that beyond a certain elongation a single R-T mode cannot generate shear in any form. Numerical results from a study of the Boussinesq equations will be presented as validation of the expanded model. Finally, the vorticity dynamics inferred from the two models is examined. The failure of the H-K model to meet the above-mentioned conservation condition in the viscous case results in the H-K model requiring a continuous viscous interaction with the rigid boundaries in order to sustain its steady state.

II. MOTIVATION FOR A COMPLEMENTARY SHEAR TERM

The dimensionless Boussinesq equations for the problem described above are

$$\frac{\partial\omega}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\omega = \sigma\frac{\partial\Theta}{\partial x} + \sigma\nabla^2\omega,$$

$$\frac{\partial\Theta}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\Theta = R\frac{\partial\Psi}{\partial x} + \nabla^2\Theta, \tag{1}$$

where ω is the vorticity ($\omega = \nabla^2\Psi$), Ψ is the stream function, $\mathbf{u}$ the velocity field [$\mathbf{u} = (\partial\Psi/\partial z, -\partial\Psi/\partial x)$], Θ the temperature perturbation away from the imposed linear temperature gradient, σ is the Prandtl number ($\sigma = \nu/\kappa$), ν is the kinematic viscosity, and κ the thermal diffusion constant. Time is scaled to $\Delta T/R$, where d is the fluid depth, ΔT the imposed temperature difference, R the Rayleigh number, and $R = \Delta T g a d^3/\nu\kappa$, where a is the thermal expansion coefficient and g is the acceleration due to gravity. The boundary conditions are that Ψ and Θ are periodic in x with period $2d/\alpha$ and that the z boundaries are *free slip*, $\Theta = \Psi = \nabla^2\Psi = 0$ for $z = 0$ and $z = d$. We will take $d = \pi$.

If we integrate the vorticity equation in (1) over x, utilizing the periodic boundary conditions and performing an integration by parts on the convection term, we obtain [6]

$$\frac{\partial}{\partial t}\int_0^{\frac{2\pi}{\alpha}}\omega dx + \frac{\partial}{\partial z}\int_0^{\frac{2\pi}{\alpha}}\nu\omega dx = \sigma\frac{\partial^2}{\partial^2 z}\int_0^{\frac{2\pi}{\alpha}}\omega dx, \tag{2}$$

where v is the z component of the velocity field. We can likewise integrate over z to obtain the following relation between the time rate of change of total vorticity and the total net viscous force between the upper and lower boundaries:

$$\frac{\partial}{\partial t}\int_0^{\pi}\int_0^{\frac{2\pi}{\alpha}}\omega dx dz = \sigma\left[\frac{\partial}{\partial z}\int_0^{\frac{2\pi}{\alpha}}\omega dx\right]_{z=0}^{z=\pi}. \tag{3}$$

Equation (3) is a fundamental conservation relation for the vorticity and, indeed, for an inviscid fluid it reduces to the conservation of total vorticity. But its derivation here did not rely on any assumptions beyond those in the Boussinesq approximation. Therefore any model that intends to capture the behavior of the vorticity equations should be consistent with it.

The original H-K model represented Ψ and Θ with the following truncated Fourier series:

$$\Psi = A\sin(\alpha x)\sin(z) + B\sin(z) + C\cos(\alpha x)\sin(2z),$$
$$\Theta = D\cos(\alpha x)\sin(z) + E\sin(2z) + F\sin(\alpha x)\sin(2z). \tag{4}$$

So ω becomes

$$\omega = -[(1+\alpha^2)A\sin(\alpha x)\sin(z) + B\sin(z) + (4+\alpha^2)C\cos(\alpha x)\sin(2z)]. \tag{5}$$

There are two cases to consider. First, when $\sigma = 0$, Eq. (3) reduces to

$$\frac{\partial}{\partial t}\int_0^{\pi}\int_0^{\frac{2\pi}{\alpha}}\omega dx dz = \frac{\partial}{\partial t}\frac{4\pi}{\alpha}B = 0, \tag{6}$$

which vanishes only if the shear mode has a zero growth rate. Likewise, if $\sigma \neq 0$ then (3) in the steady state becomes

$$\left[\frac{\partial}{\partial z}\int_0^{\frac{2\pi}{\alpha}}\omega dx\right]_0^{\pi} = 4\pi B = 0, \tag{7}$$

which requires that shear is absent, i.e., $B = 0$.

Any other consideration notwithstanding, a solution of the H-K model in the inviscid case which results in a growing shear mode or in the viscous case which leads to a steady state shear mode will be inconsistent with the physics predicted by Boussinesq equation for vorticity as stated in (1). One is motivated then to consider adding a term to the stream function complementary to $\sin(z)$.

As we will demonstrate, adding the term for the next higher shear mode as suggested by Drake *et al.* [3], $\sin(3z)$, maintains the essential character of the H-K equations while more faithfully representing the underlying equations governing the evolution of vorticity. We will show that the end result of adding the $\sin(3z)$ term is that the marginal stability boundary for the shear mode in a region of α, R, σ parameter space is significantly altered from that predicted by the H-K model.

III. THE MODIFIED H-K EQUATIONS

Consider the H-K model with the proposed additional shear term

$$\Psi = A\sin(\alpha x)\sin(z) + B\sin(z) + C\cos(\alpha x)\sin(2z) + G\sin(3z),$$

$$\Theta = D\cos(\alpha x)\sin(z) + E\sin(2z) + F\sin(\alpha x)\sin(2z). \qquad (8)$$

Substituting (8) into (1) leads to the following equations for the Fourier coefficients:

$$\frac{\partial A}{\partial t} = -\sigma(1+\alpha^2)A + \frac{\alpha\sigma D}{(1+\alpha^2)} + \frac{\alpha(3+\alpha^2)BC}{2(1+\alpha^2)} - \frac{3\alpha(\alpha^2-5)CG}{2(1+\alpha^2)},$$

$$\frac{\partial B}{\partial t} = -\sigma B - \frac{3\alpha AC}{4},$$

$$\frac{\partial C}{\partial t} = -\sigma(4+\alpha^2)C - \frac{\alpha\sigma F}{(4+\alpha^2)} - \frac{\alpha^3 AB}{2(4+\alpha^2)} + \frac{3\alpha(\alpha^2-8)AG}{2(4+\alpha^2)},$$

$$\frac{\partial G}{\partial t} = -9\sigma G + \frac{\alpha AC}{4},$$

$$\frac{\partial D}{\partial t} = -(1+\alpha^2)D + \alpha RA - \alpha AE - \frac{\alpha BF}{2} + \frac{3\alpha FG}{2},$$

$$\frac{\partial E}{\partial t} = -4E + \frac{\alpha AD}{2},$$

$$\frac{\partial F}{\partial t} = -(4+\alpha^2)F - \alpha RC + \frac{\alpha BD}{2} - \frac{3\alpha DG}{2}. \qquad (9)$$

The addition of the $\sin(3z)$ shear mode results in a contribution to the equations for D and F that is opposite in sign to the contribution arising from the $\sin(z)$ term. However, the G terms in the A and C equations have their sign dependent on α. It is the G term in the C equation that alters the behavior of the shear stability boundary.

If this enhanced model is to be an improvement on the original H-K system, it should meet the condition stated in Eq. (3). Observe that

$$\left[\frac{\partial}{\partial z}\int_0^{\frac{2\pi}{\alpha}} \omega dx\right]_0^{\pi} = -2\pi[B\cos(z) + 27G\cos(3z)]_0^{\pi} = 4\pi[B + 27G]. \qquad (10)$$

The steady state condition for Eq. (3) requires that $G = -B/27$ which is identically what one obtains from solving the system (9). Augmenting the H-K model with the $\sin(3z)$ term therefore gives us an improved model, i.e., one that is now consistent with the prediction of the behavior of total vorticity.

Linearizing Eqs. (9) about the $A = B = C = D = E = F = G = 0$ critical point leads to an eigenvalue problem that is separable into two disjoint subspaces: that spanned by the so-called Lorenz components with amplitudes A, D, and E, and that spanned by the remaining terms. There are two possible bifurcations that arise in the Lorenz space. One occurs at $R_2 = (4+\alpha^2)^3/\alpha^2$, which corresponds to the appearance of the second vertical mode of the R-T instability. This mode is incompletely represented in the model and not relevant to this study. The second bifurcation is a pitchfork type at $R_c = (1+\alpha^2)^3/\alpha^2$, which heralds the first horizontal mode of the R-T instability. The amplitudes for this solution are

$$A = \pm\frac{2\sqrt{2}}{(1+\alpha^2)}\sqrt{R-R_c},$$

$$D = \pm\frac{2\sqrt{2}(1+\alpha^2)}{\alpha}\sqrt{R-R_c}, \qquad (11)$$

$$E = R - R_C,$$

where the upper or lower sign determines the handedness of the convective cells.

Linearizing Eqs. (9) about these solutions with the non-Lorenz amplitudes set to zero again results in a decoupled problem. The solution confined to the Lorenz subspace can experience a Hopf bifurcation at $R_E = R_c\sigma[\sigma + 4/(1+\alpha^2) + 3]/[\sigma - 4/(1+\alpha^2) - 1]$ provided $R_E > R_c$. This condition requires that $\sigma > 1+4/(1+\alpha^2)$, limiting its occurrence to a region of parameter space about which we will not be concerned.

The results thus far are the same as obtained with the H-K model. However, the $\sin(3z)$ term *does* play a role in the pitchfork bifurcation that takes solution (11) out of the Lorenz subspace and leads to nonzero shear. The eigenvalue problem in the non-Lorenz subspace results in a quartic equation that yields a zero eigenvalue if $R = R^*$, where

$$\frac{R^*}{R_c} = \frac{\sigma^2\frac{(4+\alpha^2)^3}{(1+\alpha^2)^3} + \sigma\frac{10}{3} + \frac{2(4+\alpha^2)(5\alpha^2-4)}{3(1+\alpha^2)^2}}{\sigma^2 + \sigma\frac{10}{3} + \frac{2(4+\alpha^2)(5\alpha^2-4)}{3(1+\alpha^2)^2}}. \qquad (12)$$

Contrasting (12) to the result based on the H-K model, we see that the coefficient of σ is 10/3 versus 3 and the σ independent terms contain $2(5\alpha^2-4)/3$ as a factor versus α^2. For large values of α, R^* becomes dependent only on σ and (12) converges to the H-K result of 1 as $\alpha \to \infty$. For large values of σ, the constant term where the minus sign has been introduced becomes unimportant and the behavior of R^* is unaffected by the $\sin(3z)$ term over most of the range of α. However, for α and σ of order unity, dramatic differences can appear as a consequence of the fact that the denominator of R^* can now vanish.

Since $R^* \propto R_c$, $R^* \to \infty$ as $\alpha \to 0$ in both the H-K and modified H-K results. However, now asymptotes can develop for $\alpha > 0$. These values, $\alpha_{(\sigma)}$, are cutoff values for the steady shear state since, for values of $\alpha < \alpha_s(\sigma)$, $R^*/R_c < 1$ and the bifurcation to shear arising out of the R-T state no longer takes place. Figure 2 shows a plot of $\alpha_s(\sigma)$ and indicates that cutoff values exist for

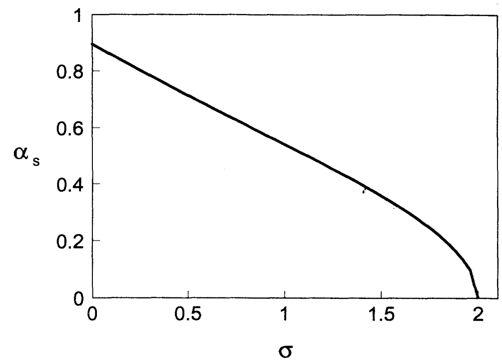

FIG. 2. Shear cutoff values α_s as a function of σ.

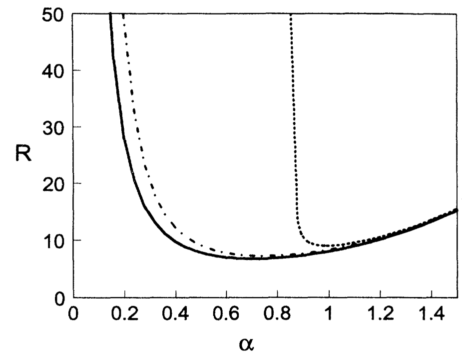

FIG. 4. R_c (solid) and R^* (dots) for the modified H-K model and the basic H-K model (dot-dash) for $\sigma = 0.1$.

$0 \leq \sigma \leq 2$. In the limit $\sigma \to 0$, $\alpha_s = \sqrt{0.8}$. Finn *et al.* [4] arrived at this same result via a perturbation analysis of the linearized, isothermal, viscous Navier-Stokes equation and using a four-component model similar to (8) for the stream function. For the sake of completeness, we note here that the cases $\sigma = 0$ and $\sigma \to 0$ produce quite different results, as observed by Finn and his co-workers. In the case $\sigma = 0$, the characteristic equation for the non-Lorenz, linear subspace of (9) yields a zero eigenvalue for all values of α. However, a second eigenvalue passes through zero at $\alpha = 2$ leading to a growing shear mode for $\alpha > 2$.

Comparative values for R_c and R^* for the H-K and the enhanced H-K models are shown in Figs. 3 and 4 for $\sigma = 1$ and $\sigma = 0.1$. The asymptotes for R^* here occur at $\alpha_s = 0.54$ and $\alpha_s = 0.86$, respectively. The additional shear term appears to only have the effect of shifting the R^* curve for $\sigma = 1$, but the presence of a cutoff value is very evident for $\sigma = 0.1$. As σ increases, the R^* curve shifts upward and the vertical asymptote moves towards $\alpha = 0$. As σ decreases, the R^* curve moves downward towards the limit of R_c while the vertical asymptote moves right toward the limiting value $\alpha_s = \sqrt{0.8}$.

It was observed in numerical studies conducted by Drake *et al.* [3] and Finn *et al.* [4] that the growth rate for the shear mode had a viscous dependence for moderate α

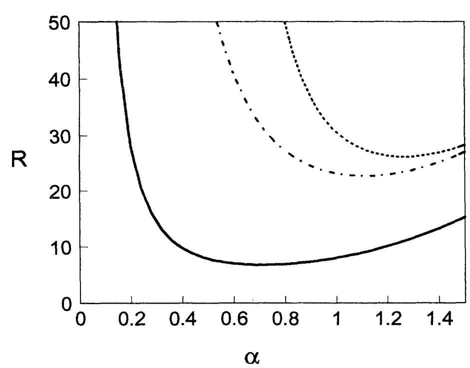

FIG. 3. R_c (solid) and R^* (dots) for the modified H-K model and the basic H-K model (dot-dash) for $\sigma = 1$.

(≈ 2) while being inviscid for larger α (≈ 4). These observations can be interpreted in terms of the dynamics of the R^* curve as σ varies. In their work, R was fixed. As σ increases, the R^* curve will approach that fixed value of R from below and eventually suppress the shear mode. As σ decreases, a larger α mode will enjoy its growing distance from the R^* curve as that stability boundary falls to R_c, increasing that mode's shear growth rate. A smaller α mode, on the other hand, cannot sustain an increasing growth rate since as the R^* curve falls away the asymptote at α_s moves in from the right, approaching that smaller α value.

IV. NUMERICAL STUDY

To validate the predictions from our modified H-K system of equations we have numerically solved the original dimensionless Boussinesq system [Eq. (1)] in two dimensions. This original set of equations is a very high-order system since the number of grid points used in the x and z directions was typically $n_x = 31$ and $n_z = 31$, respectively.

The objective of our numerical study of the Boussinesq equations was to confirm the existence of a nonzero cutoff value for α whose existence is predicted in our modified H-K model. To this end we sought to isolate this effect in parameter space, as far as possible away from the onset of higher-order horizontal modes. We chose a value of $\sigma = 0.1$ and the region near $\alpha = 1$ since here there appears a large region where the H-K model predicts steady shear, the modified model predicts no shear, and only the first horizontal mode (α mode) is present. Figure 5 depicts the curves for the critical values for the onset of the first and second horizontal modes (2α mode) and the α-mode steady shear state along with the H-K value of R^*. There is a triangular area bounded by the 2α mode on the left, the α-mode shear on the right, and the α mode below. Here the modified H-K model predicts only the α mode and no shear while the H-K model would predict shear arising from the α mode throughout this region. We did not attempt to rigorously determine the precise location of the stability boundaries but we sought to qualitatively demonstrate that the asymptotes

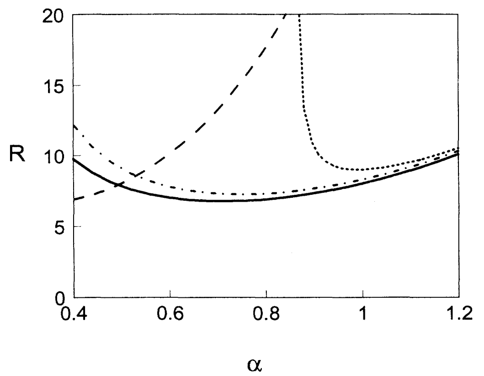

FIG. 5. R_c for the α mode (solid) and 2α mode (dashes) and R^* for the modified H-K model (dots) and the basic H-K model (dot-dash) for $\sigma = 0.1$.

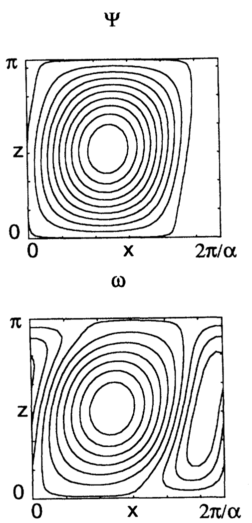

FIG. 6. Contour plot of the stream function Ψ and the vorticity ω for $\sigma = 0.1$, $\alpha = 1.1$, $R = 15$, $t = 500$.

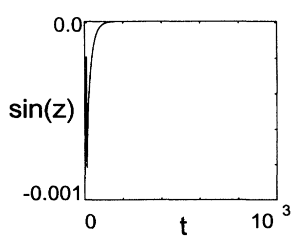

FIG. 7. Plot of the amplitude of the sin(z) Fourier component of the vorticity ω versus time for $\sigma = 0.1$, $\alpha = 1$, $R = 50$.

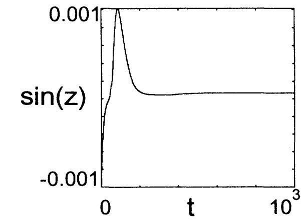

FIG. 8. Plot of the amplitude of the sin(z) Fourier component of the vorticity ω versus time for $\sigma = 0.1$, $\alpha = 0.4$, $R = 15$.

predicted by the modified H-K system are present.

We numerically integrated the system of equations (1) utilizing a leap-frog trapezoidal finite differencing scheme [10] in conjunction with a hyperviscosity term to stabilize the code [11]. Runs were made along $\alpha = 1$ and $\alpha = 1.1$ over the range $8.5 \leq R \leq 50$. $R = 8.5$ is below the critical value for the R-T mode at $\alpha = 1.1$ and for that value all initial perturbations were found to damp monotonically. Shear begins to appear at $R = 9.5$ and by $R = 15$ is strongly present as seen in the contours for the stream and vorticity functions as shown in Fig. 6. $R = 50$ is actually above the critical value for the 2α-mode but the first mode is seen to dominate. For a slightly elongated aspect ratio, $\alpha = 1$, the small shear perturbation used to seed the runs is fully damped within a single oscillation for R values up to 50 as can be seen in Fig. 7.

Next, R was fixed at a value of 15 and runs were made at α values of 0.9, 0.8, 0.5, and 0.4. The initial shear perturbation damped fully through $\alpha = 0.5$ and only the first R-T mode appears. At $\alpha = 0.4$, the 2α mode goes unstable and dominates the α mode. Since we are above the R^* value for the 2α mode, shear appears as seen in Fig. 8.

V. COMPARISON OF VORTICITY DYNAMICS BETWEEN H-K AND ENHANCED H-K MODELS

We saw in Sec. II that the original H-K model was not consistent with a prediction arising from the vorticity equation in *any* domain of parameter space. One would then expect that the physics inferred from the H-K model may be suspect. Let us compare the steady state behavior of the vorticity as predicted by H-K and modified H-K models. We will use the notation

$$\langle \cdots \rangle = \frac{\alpha}{2\pi} \int_0^{\frac{2\pi}{\alpha}} \cdots dx \tag{13}$$

to indicate an x-averaged value and we will take the steady state values of the Fourier coefficients from the solution of Eqs. (9).

We begin by observing that in Fig. 9 the shape of the equilibrium distribution of $\langle \omega \rangle$ as predicted by the two models differs substantially. Here, at steady state,

$$\langle \omega \rangle = -\alpha B[\sin(z) - \sin(3z)/3],$$

$$\langle \omega \rangle_{\mathrm{HK}} = -\alpha B_{\mathrm{HK}} \sin(z). \tag{14}$$

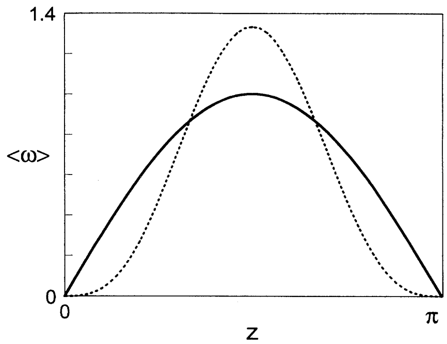

FIG. 9. x-averaged vorticity $\langle\omega\rangle$ at equilibrium for the modified H-K model (dots) and the basic H-K model (solid).

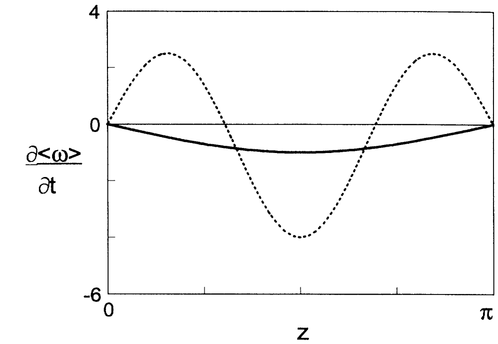

FIG. 11. Diffusive (solid) and convective (dots) contributions to the x-averaged vorticity at equilibrium for the basic H-K model.

We have normalized the absolute value of the amplitude in each case to 1 since it is the behavior of the sinusoidal shaping factor in which we are interested. Note that the sign of the vorticity could be either + or − and without loss of generality it is shown as positive with all of the following figures consistent with this choice of sign. The modified H-K model predicts the development of a boundary layer of low vorticity at the hard boundaries while the H-K model has a nonzero z derivative of $\langle\omega\rangle$ right up to the wall. Figure 10 is a diagnostic plot of $\langle\omega\rangle$ as a function of z taken from one of the numerical simulations. It clearly confirms the existence of the predicted boundary layer. We should mention that another diagnostic which computed the right-hand side of Eq. (3) turned out to be an excellent indicator of the steady state during our runs.

Equation (3) requires that the z derivative of $\langle\omega\rangle$ either vanish at each of the hard boundaries or have the same nonzero value at each wall. The modified H-K model meets the criteria by having a vanishing derivative at the walls. The H-K model fails because it has nonzero derivatives of equal *absolute* value but opposite sign at the walls. The only remaining alternative would be a model that meets condition (3) by having nonzero derivatives at the walls that sum to zero, and that would require that $\langle\omega\rangle$ vanish somewhere in the interior of $0 < z < \pi$ with $\langle\omega\rangle$ having opposite signs in the vicinity of opposite walls.

The fact that the net viscous force transferred to a wall along the x direction is $\sigma\partial\langle\omega\rangle/\partial z$ indicates that in achieving an equilibrium state the H-K model relies on continuous viscous interaction with the walls. Observe that Eq. (2) can equivalently be written as

$$\frac{\partial}{\partial t}\langle\omega\rangle = -\frac{\partial}{\partial z}\langle v\omega\rangle + \sigma\frac{\partial^2}{\partial z^2}\langle\omega\rangle. \tag{15}$$

The convective term is

$$\langle v\omega\rangle = \frac{3\alpha^2 AC}{4}[\cos(z) - \cos(3z)] \tag{16}$$

for both the original and modified H-K models. Figures 11 and 12 show the convective and diffusive contributions to the steady state for the two models, where again we have normalized the amplitude to 1. In the modified H-K model each mechanism is conservative over all z, that is, the signed area under each curve sums to zero. On the other hand, since the H-K model does not satisfy (15) in the steady state, it requires an anomalous mechanism to achieve it, to wit, the viscous interaction with the wall serving as a steady vorticity sink. This can be seen in Fig. 11. Here, the convective contribution sums to zero while the net diffusive contribution is negative.

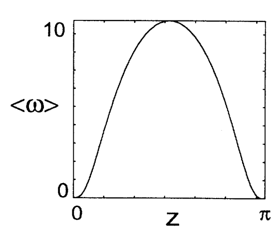

FIG. 10. x-averaged vorticity $\langle\omega\rangle$ for $\sigma = 0.1$, $\alpha = 1.1$, $R = 50$.

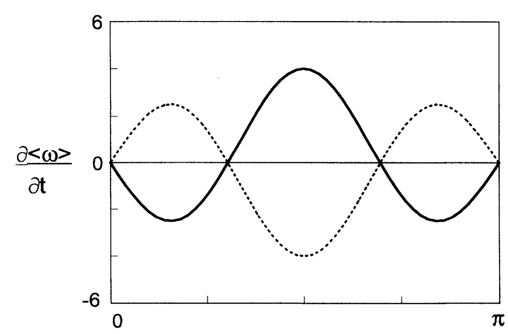

FIG. 12. Diffusive (solid) and convective (dots) contributions to the x-averaged vorticity at equilibrium for the modified H-K model.

VI. SUMMARY

The original truncated Fourier model as proposed by Howard and Krishnamurti for studying the two-dimensional shear problem has served admirably and the purpose here is not to denigrate that contribution. Recent numerical studies indicate that the inclusion of the additional $\sin(3z)$ shear term is required if the H-K model is to match observed behavior. It is therefore important to fully appreciate the limitations of the H-K model absent this term.

First, and perhaps foremost, the original H-K model does not meet a basic steady state criterion derived directly from the Boussinesq equation for vorticity. The result is that the physics inferred from the H-K model is degenerate. This degeneracy exists for the entire parameter domain. As we saw in Sec. VI, the H-K model requires that the rigid boundaries act as a constant vorticity sink at steady state. The modified model with the additional shear term has no such degeneracy.

Adding the higher-order shear mode term to the H-K system has a dramatic effect on the stability boundary for the steady shear state, although admittedly the drama is limited to a restricted region of parameter space. However, a researcher embarking on the study of shear flow using the H-K model with small α should be aware that neglecting the $\sin(3z)$ shear term may seriously affect the results. For low R we have shown that the impact of the $\sin(3z)$ term occurs well before the onset of the next higher horizontal mode.

Awareness of the above limitations of the basic H-K model will hopefully serve to guide its users to an appropriate domain of validity and alert them to the implicit physical process that it embodies.

ACKNOWLEDGMENT

This work was supported by the U.S. Department of Energy.

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